Analytical Solution for Exit gradient Variation Downstream of Inclined Sheet Pile

Prof. Dr. Rafa H. Al-Suhaili

An analytical solution for the exit gradient variation downstream of an inclined sheet pile is found. The solution is developed using the Shwarz-Christoffel transformation. The effect of angle of inclination of the sheet pile with the downstream side is investigated. Results indicate that the exit gradient is decreased as the angle is increased. The length of protection against piping is also investigated. The results indicate low variation of this protection length with the angle of inclination. The required length of protection is minimum for an angle of 5 pi/6.

Introduction

Water retaining structures such as dams or sheet piles are usually used in practice. Dams foundation is usually ended by a sheet pile at the downstream side, and almost always with an additional one at the upstream side. The downstream sheet pile is usually used to decrease the exit gradient at the downstream side of the structure in order to reduce the risk of soil piping in this side. Even though the existence of this sheet pile reduce the exit gradient, the risk of piping is still significant, hence a usual practice is to adopt a certain protection at the downstream side of the structure. This protection consists of an impervious layer, such as, slab pavement, or any kind of riprap protection with sand gravel filter. The length of protection required is usually decided upon the desired factor of safety against piping. This factor of safety is the ratio between critical exit gradient to real exit gradient. The critical exit gradient is approximated to unity Harr (1962), hence the factor of safety against piping is simply the reciprocal of the exit gradient.

From this fact if one obtains the exit gradient variation along the downstream side, the length of the required protection could be found simply for a specified factor of safety. This factor of safety depends on the soil type.

Many authors had presented formulas for obtaining the exit gradient at the toe of the structure, regardless of it's variation along the downstream side, Koslah(1954), Kochina (1952), Pavlovsky (1956) and Karroufa (1964).

Graphical flow net was used approximately for obtaining the length of protection required, Karroufa (1964). The first analytical solution obtained for exit gradient along the downstream side of the hydraulic structure was obtained by Al-Suhaili *et al*, (1988). Equations were developed for a dam with single sheet pile. The location and length of the sheet pile was variable, so many different configurations could be adopted. Al-Suhaili and Al-Kadhi(1989) had developed an analytical solutions for depressed dams without sheet piles, and another one for a depressed dam with two equal length sheet piles one at the upstream side and another at the downstream side. All of the above works were obtained for vertical sheet piles. In this research an attempt were adopted to obtain the exit gradient variation along the downstream side for an inclined sheet pile. The solution was obtained in order to find the effect of the angle of inclination of the sheet pile on the exit gradient variation and hence the required length of protection. This case is of interest not only to find this effect, but also for investigating the exit gradient variation of vertical sheet piles in anisotropic soils.

The Developed Analytical Solution

The mapping of a domain with radial slits onto an auxiliary upper half plane was first suggested by Verigin as mentioned by Harr(1964). The following transformation is applicable:

$$Z = Ce^{i\pi\alpha n}(a_1 - t)^{\alpha n - a_1}(a_2 - t)^{\alpha_1 - a_2} \ldots (ak - t)^{\alpha_{n-1} - a_n} \quad \ldots\ldots(1)$$

which maps a region with radii slits in the z-plane onto the upper half of the t-plane Fig. (1), where c is a real constant, $\pi\alpha_1$, $\pi\alpha_2$, $\pi\alpha_3$,… are the angles that the sides of the slits makes with the abscissa, $\pi\alpha_n$ is the particular angle of the slit ANFE, a_1,a_2,a_3,…,an are the image points on the real axis of the t-plane, and the point N has it's image at $t = \infty$.

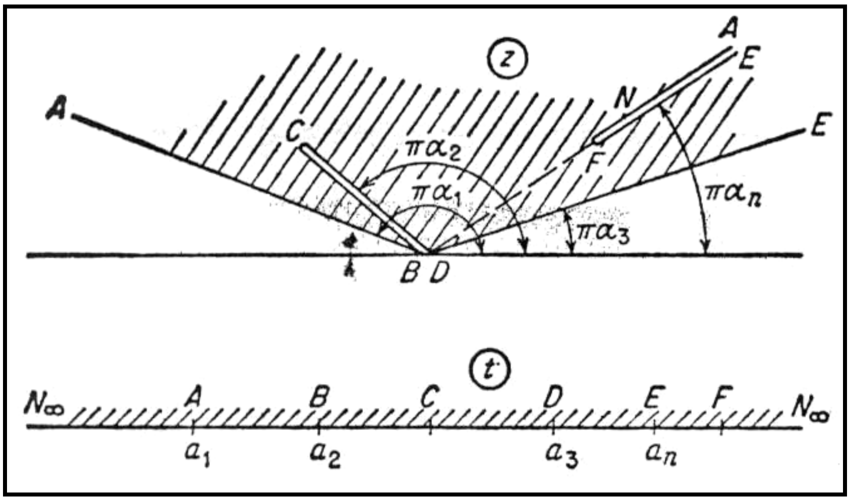


Fig. (1). Radial Slits Mapping

In order to find the exit gradient variation for a single inclined sheet pile as shown in the z-plane of Figure (2), one slit BCD is existing as in Fig. (2-b). applying the Shwarz-Christoffel transformation, we have for the mapping of the region A_∞ BCDE_∞ on to the lower half of the t-plane (Fig. 2-c), with the points B, C and D going into the points t = 1, t = a and t = +1.

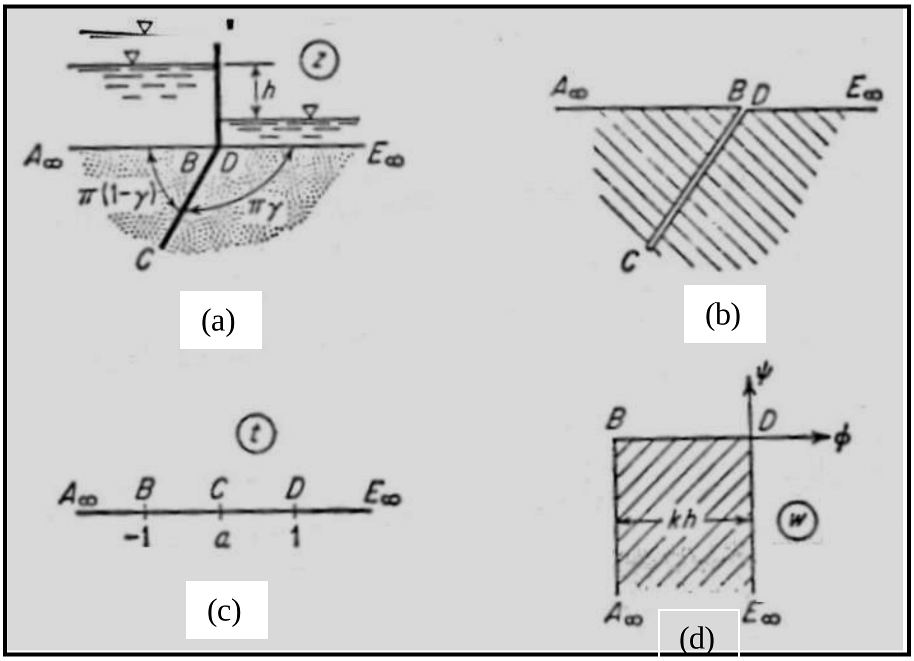


Fig. (2) Shwarz-Christoffel Transformation

$$Z = M\int \frac{dt}{(t+1)^{\gamma}\,(t-a)^{-1}\,(t-1)^{1-\gamma}} + N \text{ ....(2)}$$

and Eq. (1) for this case simply becomes :

$$Z = C(1+t)^{1-\gamma}\ (1-t)^{\gamma} \text{ ....(3)}$$

Since Equations. (2), and (3) must be equal, it follows that after differentiation :

$$\frac{dZ}{dt} = M_1(1+t)^{-\gamma}(t-a)(1-t)^{\gamma-1}$$

$$= -C(1+t)^{-\gamma}(1-t)^{\gamma-1}\,(t+2\gamma-1)$$

hence ,$M_1 = -C$, and $a = 1-2\gamma$ ....(4)

Taking s as the length of the sheet pile and noting that its tip correspond to t = a , we have

$(Z = S\,e)^{-\pi\gamma i}$ which after substituting into Equation (3), yields :

$$C = \frac{Se^{-\pi\gamma i}}{(1+a)^{1-\gamma}\ (1-a)^{\gamma}}$$

Now substituting the expression for C into Equation (3), we obtain for mapping of the z-plane onto the t-plane,

$$Z = Se^{-\pi\gamma i}\left(\frac{1+t}{1+a}\right)^{1-\gamma}\left(\frac{1-t}{1-a}\right)^{\gamma} \quad \text{...(5)}$$

where ,a = 1-2γ

Since the w-plane is a semi-infinite strip Fig. (2-d), we have for mapping of the w-plane onto the t-plane

$$\frac{dw}{dt} = \frac{Kh}{\pi\sqrt{t^2-1}} \quad \text{...(6)}$$

and ,

$$t = Cos\frac{\pi\omega}{Kh} \quad \text{...(7)}$$

The exit gradient at any point downstream of the sheet pile can be given by :

$$I_x = \frac{1}{ik}\frac{dw}{dt}\cdot\frac{dt}{dz} \quad \text{..(8)}$$

Substituting for $\frac{dw}{dt}$ and $\frac{dt}{dz}$ from the proceeding developed expressions and simplifying :

$$\frac{I_x S}{h} = \frac{-1}{\pi}\frac{(1+a)^{1-\gamma}(1-a)^{\gamma}\,e^{\pi\gamma i}}{\sqrt{1-t^2}\,(1+t)^{1-\gamma}(1-t)^{\gamma}} \quad \text{...(9)}$$

where :

h : is the head difference between upstream and downstream sides of the sheet pile

πγ : angle of inclination of the sheet pile with the right horizontal side.

S : length of the sheet pile

a = 1 – 2γ

The solution required the definition of the relation between x and t. No direct expression could be obtained for t as a function of x, hence this relation is obtained for x as a function of t. This relation is obtained from equation (5) by substituting z = x + iy with y = 0 along the downstream side, as :

$$x = \mathrm{Re}\left[S e^{\pi\gamma i} \left(\frac{1+t}{2-2\gamma} \right)^{1-\gamma} \left(\frac{1-t}{2\gamma} \right)^{\gamma} \right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

Since the required solution is for the downstream side, i.e., from point D to point E_∞, the corresponding t values will be t > 1. For t = 1 at point D equation (10) gives x = 0, since the real part will be zero. If the variation of the exist gradient is required for x-value extends not more than (3s) for example one can obtain the higher limit of t-value from equation (10), by the solution of the following equation for different values of the angle of inclination of the sheet pile γ:

$$x/S = 3 = \mathrm{Re}\left[e^{-\pi\gamma i} \left(\frac{1+t}{2-2\gamma} \right)^{1-\gamma} \left(\frac{1-t}{2\gamma} \right)^{\gamma} \right] \ldots\ldots\ldots\ldots\ldots\ldots(11)$$

A mat lab program was used to obtain the upper limits for t-value for the required upper limits x/S for the calculation of the exit gradient variation. Table (1) shows those values, for different values of the angle on inclination γ. The upper t-value was calculated for x / S =(1,2,…,10)

Table (1) Corresponding upper limits for t-values to the required upper limits of x/s for exit gradient variation

Angle of Inclination $\pi\gamma$					
	$\pi/6$	**$\pi/3$**	**$\pi/2$**	**$\pi/3$**	**$\pi/6$**
$(x/S)_u$	**tu**	**tu**	**tu**	**tu**	**tu**
1	1.1	1.3	1.5	1.7	2.1
2	2.1	2.1	2.3	2.7	3.3

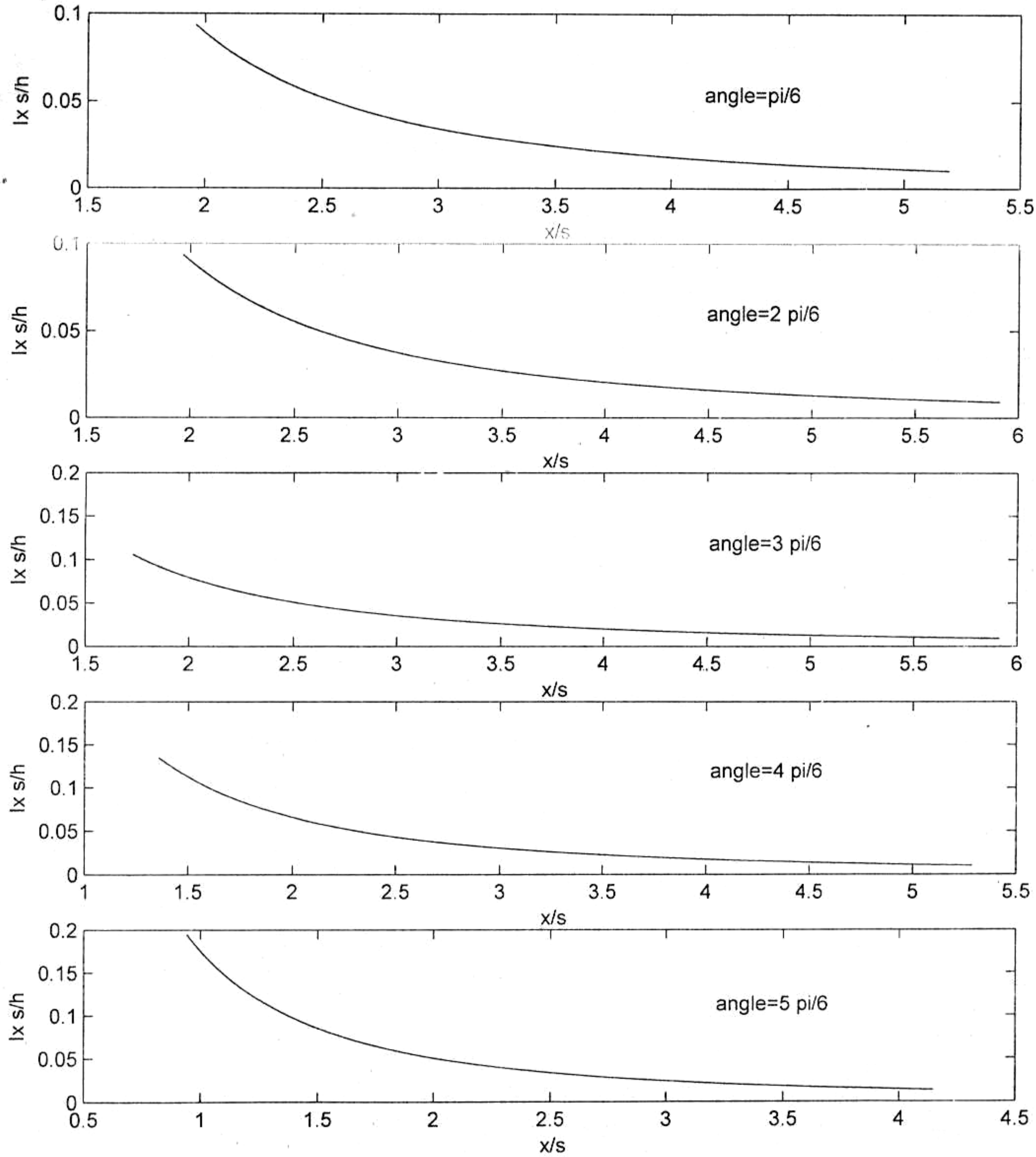
3	3.3	3	3.2	3.7	4.6
4	4.5	4.1	4.2	4.7	5.9
5	5.8	5.1	5.1	5.8	7.1
6	7.1	6.1	6.1	6.8	8.4
7	8.3	7.2	7.1	7.8	9.7
8	9.6	8.2	8.1	8.9	10.9
9	9.9	9.3	9.1	10	12.2
10	12.2	10.3	10.1	11	13.5

$(x/S)_u$: upper limit of x/s value for exit gradient variation

tu : upper limit of t-value ($1< t < tu$)

Equation (9) is used for calculating the exit gradient variation for γ = 1/6, 2/6, 3/6, 4/6 and 5/6. A mat lab program is used for this purpose. The results were shown in Fig. (3). The results indicates high variation of IxS/h for low x/S values. As the distance extends the variation is reduced and the values of IxS/h approaches zero as expected.

In order to investigate the effect of the angle of the sheet pile an illustrative example is used. Suppose that the difference in head is 30 m, and the sheet pile length is (S = 5 m). For a factor of safety against piping required as mentioned by Koslah(1954) as 5 for shingle, 6 for coarse sand and for fine sand. Table (2) shows the length of protection required for different values of the angle of inclination of the sheet pile.



Fig(3) Variation of Exit Gradient Downstream of an Inclined sheetpile.

Table (2) Required Length of Protection Against Piping Downstream of an Inclined Sheet Pile, h = 30 m, S = 5.

γ / material	1/6	2/6	3/6	4/6	5/6
Shingle	14.87	15.60	15.4	14.36	12.57
Coarse sand	16.49	17.28	16.98	15.73	13.85
Fine sand	17.69	18.65	18.33	17.1	15.04

Fig. (4) Shows the length of protection required against piping with different values of angle of inclination, for shingle, coarse sand and fine sand. The figure indicates an increase in the protection length followed by a decrease after approximately 90° angle of inclination.

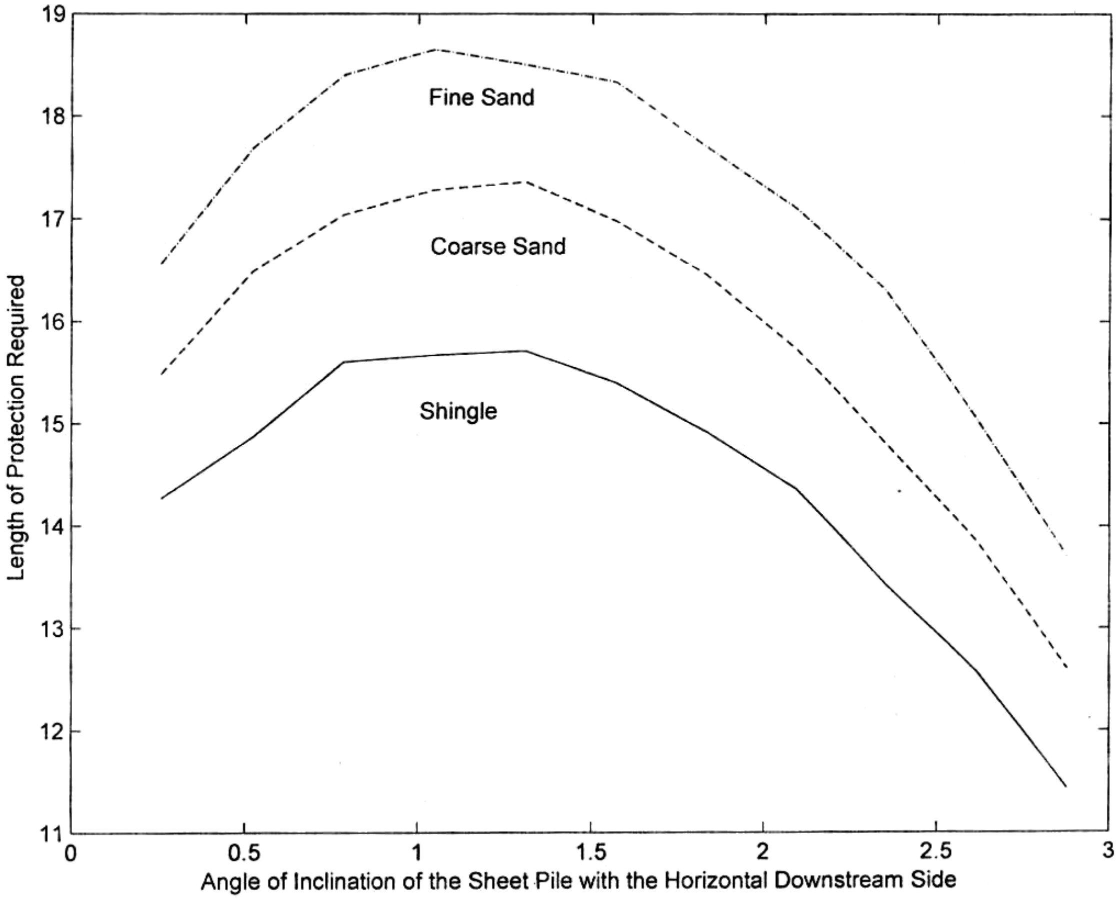


Fig. (4) Variation of the Required Protection Length against Piping with Angle of Inclination of the Sheet Pile

Conclusions

1. The exit gradient downstream of an inclined sheet pile is in general decreased with the increase of its angle of inclination measured to the horizontal line to the right of the sheet pile.

2. The required length of protection to achieve the factor of safety against piping is in general decreased with the angle of inclination increase. This behavior is observed beyond the angle of $2\pi/6$. However the variation of this length required is small for all the three types of the soils downstream of the inclined sheet pile. Moreover, when the angle of inclination more than 90° the protection length decreases with the angle of inclination increase.

References

1. Al-Douri, R.M.S. (1986); "downstream Protection of Hydraulic Structures". M.Sc. Thesis, Department of Buildings and Constructions, University of Technology Baghdad, Iraq.
2. Al-Shakarchi, Y.J. (1985); "Seepage Around Sheet Piles-Fragment Method". Iraqi Conference on Engineering, College of Engineering, University of Baghdad, Baghdad, Iraq.

3. Al-Suhaili, R.H., Kharrufa, N. S and Al-Oubaidy A. S, ”Variation of Exit Gradient Downstream of Hydraulic Structure", Engineering and Technology, Vol. 5, No. 3, 1988, PP. 117-135.

3. Behnam M.A., "Effect of Anisotropic Soil on Exit Gradient Hydraulic structures", M.Sc. Thesis, College of Eng.Univ. of Baghdad, Dept. of irrigation and Drainage Engineering., 1989.
4. Byrd. P.F. and. M.D. Fridman., "Handbook of Elliptic Integrals for Engineers and Scientists", Springer-ver1ag New York, 1971.

5. Harr, M. E. (1962); "Ground water and Seepage McGraw-Hill Co., New York, 271 p.

6. Kharrufa, M.S.; "Uplift Pressure and seepage Under Irrigation Structures", Iraqi Society of Engineers sixth Conference. Baghdad, Iraq, (1964).

7. Khosla, R.B.A.N., Boss, N.K. and Taylor E. Mck, 1954; "Design of wiers on permeable foundations", Central Board of Irrigation , New Delhi, India.

8. Mushat M., (1937); "The Flow of Homogeneous Fluids Through Porous Media", McGraw-Hill Co., New York. 1937, Reprinted in 1964.

9. Pavlovisky, N.N., (1956); "collected Works", Akad. Nauk U.S.R., Leningrad, 1956

10. Polubarinova-Kochina, P.Na, 1952; "Theory of Ground water movement", Translated by J.M. DeWiest, Princton University Press.